Virtual Classes of Character Stacks

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3 March, 2022

Representation varieties

- M = connected closed manifold
- $\pi_1(M)$ = fundamental group
- G = algebraic group over k

G-**representation variety** of M $\qquad R_G(M) = \mathsf{Hom}(\pi_1(M), G)$

Note: G acts on $R_G(M)$ by conjugation $\quad (g \cdot \rho = g\rho g^{-1})$

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Non-Abelian Hodge Theory: for smooth complex projective curve C, and $G = \mathrm{GL}_n(\mathbb{C})$,

$$\chi_G(C) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{rank } n \text{ vector bundles with} \\ \text{flat connection on } C \end{array} \right\}$$

$$\chi_G(C) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{semistable Higgs bundles on } C \\ \text{of rank } n \text{ and degree } 0 \end{array} \right\}$$

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$$[X/G](T) = \left\{ \begin{array}{ccc} P & \xrightarrow{G\text{-equiv}} & X \\ \downarrow{\scriptstyle \pi} & & \\ T & & \end{array} \right\}$$

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- G-equivariant geometry

Study by additive/motivic invariants:

$$\mu : \textbf{Var}_k (\text{or } \textbf{Stck}_k) \to R$$

such that $\mu(X) = \mu(Z) + \mu(X \setminus Z)$ for closed $Z \subset X$

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Examples

- Euler characteristic $\chi(X) \in \mathbb{Z}$,
- Points over finite fields $\#X(\mathbb{F}_q) \in \mathbb{Z}$
- Deligne–Hodge polynomial

$$e(X) = \sum_{k,p,q} (-1)^k h_c^{k;p,q}(X) u^p v^q \in \mathbb{Z}[u, v].$$

Grothendieck ring of varieties / stacks

$$K(\textbf{Var}_k) = \bigoplus_{\text{isom. cl. } [X]} \mathbb{Z} \cdot [X] \Big/ \sim$$

- (scissor) $[X] = [Z] + [X \setminus Z]$ for closed $Z \subset X$
- (multiplication) $[X] \cdot [Y] = [X \times_k Y]$
- $[\varnothing] = 0$ and $[\text{Spec}(k)] = 1$

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$$\textbf{Var}_k \longrightarrow K(\textbf{Var}_k) \xrightarrow{\chi} \mathbb{Z}, \quad K(\textbf{Var}_k) \xrightarrow{e} \mathbb{Z}[u, v], \quad K(\textbf{Var}_k) \xrightarrow{\mu} R$$

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[\mathsf{SL}_2(k)] &= \{a = 0\} &+& \quad \{a \neq 0\} \\
&= \{d \in k\} \cdot \{c = -b^{-1}\} &+& \quad \{d = (1 + bc)/a\}
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&= \{d \in k\} \cdot \{c = -b^{-1}\} + \{d = (1 + bc)/a\} \\
&= q(q-1) + q^2(q-1) \\
&= q^3 - q
\end{aligned}
$$

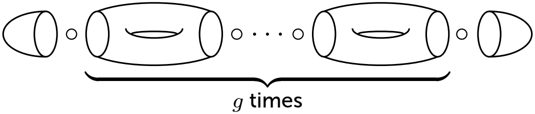
with $q = [\mathbb{A}^1_k]$

First goal: compute class of $[R_G(\Sigma_g)]$ in $K(\mathbf{Var}_k)$

Later: compute class of $[\mathfrak{X}_G(\Sigma_g)]$

Idea: cut manifold in pieces and 'compute invariant piecewise'

$\Sigma_g \;=$

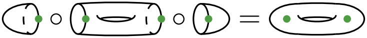


Using *Topological Quantum Field Theory*

$$Z : \mathbf{Bdp}_n \to R\text{-}\mathbf{Mod}$$

Category of (pointed) bordisms $\mathbf{Bdp}_n$

- Objects (M, A) e.g.  $\varnothing$
- Morphisms (W, A) e.g.   
- Composition: glue along common boundary

e.g. 

Define TQFT $Z : \mathbf{Bdp}_2 \to K(\mathbf{Var}_k)\text{-}\mathbf{Mod}$

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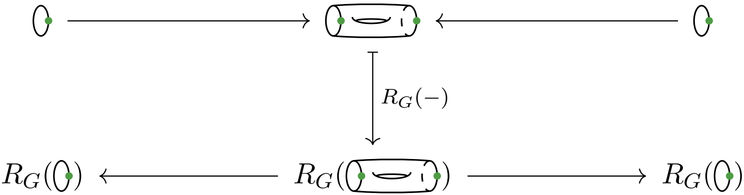
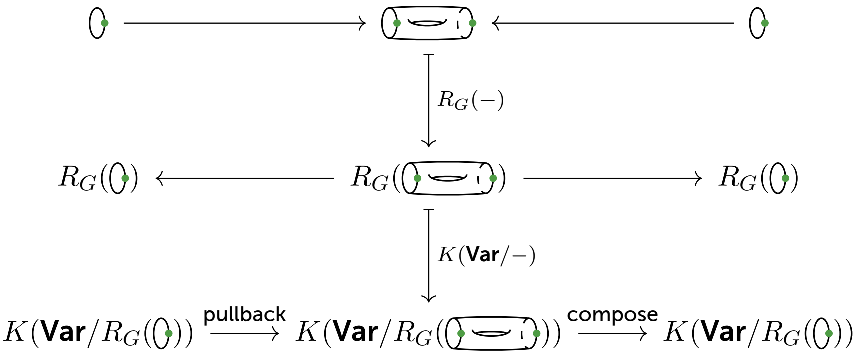


Define TQFT $Z : \mathbf{Bdp}_2 \to K(\mathbf{Var}_k)$-**Mod**



$$Z : \mathbf{Bdp}_n \to K(\mathbf{Var}_k)\text{-}\mathbf{Mod}$$

Closed surface $\Sigma_g : \varnothing \to \varnothing$, so by (lax) monoidality

$$Z(\Sigma_g) : Z(\varnothing) \to Z(\varnothing)$$

For us, $\quad Z(\Sigma_g)(1) = [R_G(\Sigma_g)] \in K(\mathbf{Var}_k)$

$$[R_G(\Sigma_g)] = \frac{1}{[G]^g} \; Z($$  $$) \circ Z($$  $$)^g \circ Z($$  $$)(1)$$

and it remains to compute $Z($  $)$

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- Current results: SL_2, $\mathbb{T}_n$ and $\mathbb{U}_n$ for $n = 2, 3, 4$ (and 5)
- Generalizations: can be extended to parabolic settings, non-orientable surfaces

Stacky TQFT

Replace $$K(\mathbf{Var}_k) \longmapsto K(\mathbf{Stck}_{\mathrm{BG}})$$

$$R_G(X) \longmapsto \mathfrak{X}_G(X)$$

to obtain $$\mathcal{Z} : \mathbf{Bdp}_n \to K(\mathbf{Stck}_{\mathrm{BG}})\textbf{-Mod}$$

$$[\mathfrak{X}_G(\Sigma_g)] = \frac{1}{[G/G]^g} \mathcal{Z}(\;\cdot\;) \circ \mathcal{Z}(\;\cdot\;)^g \circ \mathcal{Z}(\;\cdot\;)(1)$$

$$\mathsf{AGL}_1(k) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$$

Theorem [González–Prieto, Hablicsek, Vogel, arXiv:2201.08699]

The virtual class of the character stack $[\mathfrak{X}_{\mathsf{AGL}_1(k)}(\Sigma_g)]$ equals

$$\mathsf{B}G + ((q-1)^{2g} - 1)[\mathbb{A}^1_k/G] + \frac{q^{2g}-1}{q-1}[\mathbb{G}_m/G]$$

$$+ \frac{\left(q^{2g-2}-1\right)\left((q-1)^{2g}-1\right)}{q-1}[\mathsf{AGL}_1(k)/G]$$

Obtain $[R_{\mathsf{AGL}_1(k)}(\Sigma_g)]$ from $[\mathfrak{X}_G(\Sigma_g)]$ by 'forgetting G-action'

$$K(\textbf{Stck}_{\mathsf{BG}}) \to K(\textbf{Var}_k), \quad [X/G] \mapsto [X]$$

$$\mathsf{BG} \mapsto 1, \quad [\mathbb{A}^1_k/G] \mapsto q, \quad [\mathbb{G}_m/G] \mapsto q-1, \quad [\mathsf{AGL}_1(k)/G] \mapsto q(q-1)$$

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Similarly,

- for any $H \subset G$, fixed points

$$(-)^H : K(\textbf{Stck}_{\mathsf{BG}}) \to K(\textbf{Var}_k), \quad [X/G] \mapsto [X^H]$$

- free/transitive locus
- stabilizers

In particular, for $G = \mathsf{AGL}_1(k)$, obtain class GIT quotient via

$$\chi_G(\Sigma_g) \simeq (R_G(\Sigma_g))^H,$$

with $H = \{\text{diagonals}\} \subset \mathsf{AGL}_1(k)$.

Hence, $(-)^H : K(\textbf{Stck}_{\mathsf{BG}}) \to K(\textbf{Var}_k)$ with

$$\mathsf{BG} \mapsto 1, \quad [\mathbb{A}^1_k / G] \mapsto 1, \quad [\mathbb{G}_m / G] \mapsto 0, \quad [\mathsf{AGL}_1(k)/G] \mapsto 0.$$

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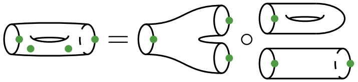

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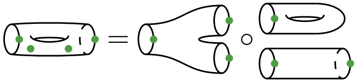
- Write 

- Advantage: can easily generalize to parabolic case

Tool 1 If $H \subset G$ subgroup, then

$$K(\mathbf{Stck}_{\mathsf{BH}}) \to K(\mathbf{Stck}_{\mathsf{BG}})$$

$$[Y/H] \mapsto [(G \times_H Y)/G]$$

is a $K(\mathbf{Stck}_k)$-module morphism.

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Example

$G = \mathbb{G}_m$ acts on $X = \mathbb{A}^1 - \{0\}$ by $t \cdot x = t^2 x$,

then $Y = \{1\}$ with $H = \mathbb{Z}/2\mathbb{Z}$.

Tool 2 Suppose

$$\begin{array}{ccccc} X & \xleftarrow{2:1} & X' & \overset{\sim}{=\!=\!=} & F \times B' \\ \downarrow & & \downarrow & & \\ B & \xleftarrow{2:1} & B' & & \end{array}$$

then $X \simeq (F \times B') /\!\!/ (\mathbb{Z}/2\mathbb{Z})$

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Theorem

$$e(X) = e(F)^{+} e(B)^{+} + e(F)^{-} e(B)^{-}$$

with $e(T)^{+} = e(T \mathbin{/\!\!/} (\mathbb{Z}/2\mathbb{Z}))$ and $e(T)^{-} = e(T) - e(T)^{+}$

For stacks,

$$[X/G]^+ = [(X /\!\!/ \mathbb{Z}/2\mathbb{Z})/G], \qquad [X/G]^- = [X/G] - [X/G]^+$$

Theorem [González–Prieto, Hablicsek, Vogel]

$$[T \times X/G]^+ = [T]^+[X/G]^+ + [T]^-[X/G]^-$$

for $[T] \in \mathcal{V}$, with $\mathcal{V} \subset K(\textbf{Stck}_{\mathsf{B}(\mathbb{Z}/2\mathbb{Z})})$ a subring, containing at least

- $T = \mathbb{P}^n$ (any $\mathbb{Z}/2\mathbb{Z}$-action)
- $T = \mathbb{Z}/2\mathbb{Z}$
- $T = B(\mathbb{Z}/2\mathbb{Z})$

Currently: working on the case $G = \mathsf{SL}_2(\mathbb{C})$ ...